

NULL PARALLEL P-EQUIDISTANT B-SCROLLS

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ABSTRACT. In this paper, null parallel p-equidistant B-scrolls are defined in 3-dimensional Minkowski space R_1^3 . We prove necessary and sufficient conditions for these B-scrolls to be equivalent of their Cartan frames. The relations between matrices of the shape operators and the algebraic invariants (Gauss, mean curvatures, principal curvatures) of these B-scrolls are shown. Besides we give the relations between second Gauss curvatures, mean curvatures and the distribution parameters of non-developable null parallel p-equidistant B-scrolls. Finally, an example is given related to the null parallel p-equidistant B-scrolls in R_1^3 .

1. BASIC CONCEPTS:

The Minkowski 3-space R_1^3 is the Euclidean 3-space R^3 provided with the standard flat metric

$$(1.1) \quad h = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of R_1^3 . Since h is an indefinite metric, recall that a vector $v \in R_1^3$ can have one of three Lorentzian causal characters: it can be spacelike if $h(v, v) > 0$ or $v = 0$, timelike if $h(v, v) < 0$ and null(lightlike), if $h(v, v) = 0$ and $v \neq 0$. The norm of a vector $v \in R_1^3$ is defined as $\|v\| = \sqrt{|h(v, v)|}$. Therefore, v is a unit vector if $h(v, v) = \pm 1$. Furthermore, vectors v and w are said to be orthogonal if $h(u, w) = 0$, [11]. For any vectors $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in R_1^3$, the Lorentzian product $v \wedge w$ of v and w is defined as [1]

$$(1.2) \quad v \wedge w = (v_2w_3 - v_3w_2, v_1w_3 - v_3w_1, v_2w_1 - v_1w_2).$$

Similarly an arbitrary curve $\alpha = \alpha(s)$ in R_1^3 can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null (lightlike), respectively. If $h(\alpha'(s), \alpha'(s)) = \pm 1$ then α is a unit speed curve and s is arc-length parameter of α , [10].

Let M be a surface in 3-dimensional Minkowski space R_1^3 . The surface M is called a timelike surface if the induced metric on the surface is a Lorentzian metric.

Therefore, the normal of this timelike surface is a spacelike vector, [4].

Let us suppose that $h(P_1, P_2) = 0$. The following table is valid for the plane π which is spanned by the vectors P_1 and P_2 according to being the vectors P_1 and P_2 spacelike, timelike or null vectors, [8].

2010 *Mathematics Subject Classification.* 53B30, 53C50.

Key words and phrases. ruled surface, Minkowski, B-Scroll, null, parallel p-equidistant.

| $\begin{array}{c} P_1 \\ \swarrow \\ P_2 \end{array}$ | spacelike | timelike | null |
|---|-----------------|----------------|------------|
| spacelike | π spacelike | π timelike | π null |
| timelike | π timelike | — | — |
| null | π null | — | — |

Table 1

Let M be a Lorentz manifold and α be a null curve on the manifold M in R_1^3 . The frame in R_1^3 is positively oriented triple (ℓ, n, u) of vectors which satisfies the following conditions

$$(1.3) \quad \begin{aligned} h(\ell, \ell) &= h(n, n) = 0, & h(\ell, n) &= -1 \\ h(\ell, u) &= h(n, u) = 0, & h(u, u) &= 1 \end{aligned}$$

A null frame for a null curve $\alpha(s)$ is a frame field $F(\ell(s), n(s), u(s))$ such that $\frac{d\alpha}{ds}$ is positive scalar multiple of ℓ , [9]. In this situation, (α, F) couple is called framed null curve. Frames of null curves are not unique. Moreover, frames are changed under reparametrization of a curve. Therefore, the curve and the frame must be given together. The Frenet formulas of α with respect to the frame F are given by [9]:

$$(1.4) \quad \begin{aligned} \frac{d\ell}{ds} &= k\ell + \kappa u, \\ \frac{dn}{ds} &= -kn + \tau u, \\ \frac{du}{ds} &= \tau\ell + \kappa n. \end{aligned}$$

The functions k , κ and τ are called the curvature functions of α . There always exists a parameter s of α such that $k = 0$ in (1.5). This parameter is called a distinguished parameter of α [6].

Then the Frenet formula of α can be written by

$$(1.5) \quad \begin{aligned} \frac{d\ell}{ds} &= \kappa u, \\ \frac{dn}{ds} &= \tau u, \\ \frac{du}{ds} &= \tau\ell + \kappa n. \end{aligned}$$

Here, ℓ is the tangent vector (rather than its direction) and u is analogous to the principal normal in the standart Frenet frame for a curve in E^3 , [9]. The null frame $F(\ell(s), n(s), u(s))$ is called Cartan frame of α . A parametrized null curve parametrized by distinguished parameter s together with its Cartan frame is called a Cartan framed null curve [6]. In addition to that

$$(1.6) \quad \ell \wedge n = u, \quad n \wedge u = -\ell, \quad \ell \wedge u = n,$$

A ruled surface is a surface swept out by a straight line Y moving along a curve α . The various positions of the generating line Y are called the rulings of the surface. Such a surface has a parametrization in ruled form as follows:

$$(1.7) \quad \varphi(s, v) = \alpha(s) + vY(s).$$

We call α to be the base curve and Y to be the director vector. If the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable surface. The remaining ruled surfaces are called skew surfaces. If there exists a common perpendicular to two preceding rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the curve of striction [4], [11].

In 3-dimensional Minkowski space R_1^3 if the base curve α and the director vector Y are chosen as a null curve and a null line, respectively, then the ruled surface is called null scroll and denoted by M . It is easily seen that the null scroll M is a timelike surface. Especially when $\kappa(s) \neq 0$ and $\tau(s) = \text{constant}$, the null scroll M is called a B-scroll and parametric equation is given as follows [7]:

$$\varphi(s, v) = \alpha(s) + vn(s)$$

B-scrolls were first introduced by Graves [5] and used to classify the codimension one isometric immersion between Lorentz surfaces. Then some authors studied and developed the geometry of B-scroll [13].

Let M be a B-scroll. Then the tangent planes along a ruling of M coincide if and only if $\tau = 0$, [3].

The striction curve and drall (distribution parameter) of non-developable B-scroll in R_1^3 , respectively, is given as follows [3]:

$$(1.8) \quad \beta(s) = \alpha(s) - \frac{h(\alpha'(s), n'(s))}{\|n'(s)\|} n(s)$$

and

$$(1.9) \quad \lambda = -\frac{\det(\ell, n, n')}{\|n'(s)\|}$$

Let M be a surface in R_1^3 . If D and N be Levi-Civita connection and the unit normal vector field on M , respectively then the shape operator of M which is obtained from N is defined by

$$(1.10) \quad S(X) = -D_X N, \quad \forall X \in \chi(M)$$

where $\chi(M)$ is the space of the vector fields of M , [11].

Let $S(P)$ is the shape operator of M at the point P . Therefore,

$$(1.11) \quad \begin{aligned} K : M &\rightarrow IR \\ P &\rightarrow K(P) = \det S(P) \end{aligned}$$

the function is called as Gaussian curvature function and also the value of $K(P)$ is the Gaussian curvature of M at the point P . Similarly

$$(1.12) \quad \begin{aligned} H : M &\rightarrow IR \\ P &\rightarrow H(P) = \frac{\text{tr} S(P)}{\text{boy} M} \end{aligned}$$

the function is called as mean curvature function and also the value of $H(P)$ is called as mean curvature of M at the point P .

If M is a non-developable surface and E, F, G are the coefficients of the first fundamental form I , then E, F, G are as follows [12]:

$$(1.13) \quad E = h(\varphi_s, \varphi_s), \quad F = h(\varphi_s, \varphi_v), \quad G = h(\varphi_v, \varphi_v).$$

If we take $D = EG - F^2$, then the coefficients of the second fundamental form e, f and g of M are given as

$$(1.14) \quad e = \frac{h(\varphi_{ss}, \varphi_s \wedge \varphi_v)}{D}, \quad f = \frac{h(\varphi_{sv}, \varphi_s \wedge \varphi_v)}{D}, \quad g = \frac{h(\varphi_{vv}, \varphi_s \wedge \varphi_v)}{D}.$$

Besides, the second Gaussian curvature of the surface M is given as, [12].

$$(1.15) \quad K_{II} = \frac{1}{(|eg| - f^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{vv} + f_{sv} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_v \\ f_v - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_v & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & \frac{1}{2}g_s \\ \frac{1}{2}e_v & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}.$$

Let M be a non-developable surface in R_1^3 . If K and H denote the mean curvature and the Gaussian curvature of the surface M , respectively, then the second fundamental form II can be written as

$$II = L_{ij} dx_i dx_j$$

So, the second mean curvature H_{II} of M is given by

$$(1.16) \quad H_{II} = H - \frac{1}{2\sqrt{|\det II|}} \sum_{ij} \frac{\partial}{\partial u^i} \left(\sqrt{|\det II|} L^{ij} \frac{\partial}{\partial u^j} (\ln \sqrt{|K|}) \right)$$

where L^{ij} is the inverse of the matrix representation L_{ij} of the second fundamental form and u^1, u^2 correspond to the parameters s and v , respectively, [12].

Throughout this paper, we consider n and α as a null vector and a null curve, respectively.

2. NULL PARALLEL p -EQUIDISTANT B-SCROLLS IN R_1^3

Let α be a null curve and $\{\ell(s), n(s), u(s)\}$ be a Cartan frame of the null curve α . If the null vector $n(s)$ moves along the null curve α , then the B-scroll is given by the following parametrization

$$(2.1) \quad \varphi(s, v) = \alpha(s) + v n(s)$$

where, the curve α and the vector $n(s)$ are called base curve and the director vector of B-scroll, respectively. This B-scroll is denoted by M ($\kappa(s) \neq 0$, $\tau(s) = \text{constant}$, see [7]). Considering equation (2.1) we have

$$(2.2) \quad \varphi_s = \ell(s) + v\tau u(s), \varphi_v = n(s)$$

Therefore, the unit normal vector N_0 of B-scroll M is

$$(2.3) \quad N_0 = \frac{\varphi_s \wedge \varphi_v}{\|\varphi_s \wedge \varphi_v\|} = u(s) + v\tau n(s).$$

It is obvious that the unit normal vector of M is space-like vector. It means that the B-scroll M is a time-like surface.

The planes corresponding to the sub-spaces $Sp\{\ell, n\}$, $Sp\{n, u\}$ and $Sp\{u, \ell\}$ along the base curve α of B-scroll M are called central plane, polar plane and asymptotic plane, respectively. From table 1, the central plane is time-like, the polar and the asymptotic planes are null planes.

Let α^* be another null curve with Cartan frame

$F^* = (\ell^*(s^*), n^*(s^*), u^*(s^*))$ so that

$$(2.4) \quad \begin{aligned} \ell^*(s^*) &= \frac{d\alpha^*}{ds^*}, & h(n^*, n^*) &= h(\ell^*, \ell^*) = 0, \\ h(\ell^*, u^*) &= h(n^*, u^*) = 0, & h(\ell^*, n^*) &= -1, & h(u^*, u^*) &= 1. \end{aligned}$$

In addition to that, Frenet formulas of α^* with respect to Cartan frame $F^*(\ell^*, n^*, u^*)$ are

$$(2.5) \quad \begin{aligned} \frac{d\ell^*}{ds^*} &= \kappa^* u^* \\ \frac{dn^*}{ds^*} &= \tau^* u^* \\ \frac{du^*}{ds^*} &= \tau^* \ell^* + \kappa^* n^* \end{aligned}$$

Let M^* be a B-scroll. Thus, the B-scroll M^* is parametrically given by as

$$(2.6) \quad \varphi^*(s^*, v^*) = \alpha^*(s^*) + v^* n^*(s^*)$$

Definition 2.1. Let M and M^* be two B-scrolls in R_1^3 and p be the distance between the asymptotic planes of M and M^* . If

- 1) The generator vectors of M and M^* are parallel,
- 2) The distance p is constant, then the pair of B-scrolls M and M^* are called the null parallel p -equidistant B-scrolls in R_1^3 .

Thus, the parametric representation of null parallel p -equidistant B-scrolls are given as follows:

$$(2.7) \quad \begin{aligned} M : \varphi(s, v) &= \alpha(s) + vn(s) \\ M^* : \varphi^*(s^*, v^*) &= \alpha^*(s^*) + v^* n^*(s^*) \end{aligned}$$

Throughout this paper, the base curves of null parallel p -equidistant B-scrolls M and M^* are also striction curves.

Theorem 2.1. Let M and M^* be two null parallel p -equidistant B-scrolls in R_1^3 . The Cartan frames $\{\ell, n, u\}$ and $\{\ell^*, n^*, u^*\}$ of the base curves $\alpha(s)$ and $\alpha^*(s^*)$ are equivalent at the corresponding points in M and M^* , respectively if and only if $\tau^* = \tau \frac{ds}{ds^*}$ and $\kappa^* = \kappa \frac{ds}{ds^*}$.

Proof: Firstly, suppose that the Cartan frames $\{\ell, n, u\}$ and $\{\ell^*, n^*, u^*\}$ of base curves of $\alpha(s)$ and $\alpha^*(s^*)$ are equivalent at the corresponding points of M and M^* , respectively.

This means that

$$(2.8) \quad \ell = \ell^*, \quad n = n^*, \quad u = u^*$$

From equation (2.5) we reach

$$\kappa^* = h\left(\frac{d\ell^*}{ds^*}, u^*\right).$$

Considering the last equation together with hypothesis we easily see that

$$(2.9) \quad \kappa^* = \kappa \left(\frac{ds}{ds^*}\right)$$

Using equation (2.5) and following similar way, we get

$$(2.10) \quad \tau^* = \tau \frac{ds}{ds^*}$$

Conversely, let the relationship between the curvature and torsion of M and M^* be $\kappa^* = \kappa \frac{ds}{ds^*}$ and $\tau^* = \tau \frac{ds}{ds^*}$, respectively.

Since M and M^* are null parallel p -equidistant B-scrolls, the generator vectors n and n^* of M and M^* , respectively are parallel. Therefore, we can choose

$$(2.11) \quad n^* = n.$$

From the last equation, we have

$$(2.12) \quad \frac{dn}{ds} = \frac{dn^*}{ds^*} \cdot \frac{ds^*}{ds}$$

Substituting the equations (1.6) and (2.5) into the last equation, by routine calculation, one can obtain

$$u^* = u$$

In addition to that, we can write from the last equation

$$(2.13) \quad \frac{du}{ds} = \frac{du^*}{ds^*} \cdot \frac{ds^*}{ds}$$

Using the equation (1.6) and (2.5) together with the equation (2.13), it is seen that

$$\ell^* = \ell.$$

This completes the proof.

Theorem 2.2. *In R_1^3 , let M and M^* be null parallel p -equidistant B-scrolls, S and S^* be the matrices corresponding to the shape operators of M and M^* , respectively. Then, there is a relation between the matrices S and S^* as follows*

$$S^* = \left(\frac{ds}{ds^*} \right) S.$$

Proof: Let us find the matrices of S and S^* , which is correspond to shape operators of null parallel p -equidistant B-scrolls M and M^* , respectively. From equations (1.6) and (2.3), we obtain

$$(2.14) \quad S(\varphi_v) = -D_{\varphi_v} N_0 = -\frac{dN_0}{dv} = -\tau n(s)$$

and

$$(2.15) \quad S(\varphi_s) = -D_{\varphi_s} N_0 = -\frac{dN_0}{ds} = -\tau \ell(s) - \kappa(s)n(s) - v\tau^2 u(s).$$

Considering the equations (2.14) and (2.15) together with the equation (2.2), we reach

$$S(\varphi_v) = -\tau \varphi_v + 0 \varphi_s$$

and

$$S(\varphi_s) = -\kappa(s) \varphi_v - \tau \varphi_s.$$

In this case, the matrix which is correspond to shape operators of M as follows,

$$(2.16) \quad S = \begin{bmatrix} -\tau & 0 \\ -\kappa(s) & -\tau \end{bmatrix}.$$

Using the equations (2.3) and (2.5) and following similar way, we can get

$$S^*(\varphi_{v^*}^*) = -\tau^* \varphi_{v^*}^* - 0 \varphi_{s^*}^*$$

and

$$S^*(\varphi_{s^*}^*) = -\kappa^*(s^*)\varphi_{v^*}^* - \tau^*\varphi_{s^*}^*$$

Thus, the matrix corresponding to shape operator of M^* is

$$S^* = \begin{bmatrix} \tau^* & 0 \\ -\kappa^*(s^*) & -\tau^* \end{bmatrix}.$$

Substituting the equations (2.9) and (2.10) into the last equation, we obtain

$$(2.17) \quad S^* = \begin{bmatrix} \left(\frac{ds}{ds^*}\right)\tau & 0 \\ -\left(\frac{ds}{ds^*}\right)\kappa(s) & -\left(\frac{ds}{ds^*}\right)\tau \end{bmatrix}.$$

From the equations (2.16) and (2.17), it is obvious that

$$(2.18) \quad S^* = \left(\frac{ds}{ds^*}\right) S.$$

Theorem 2.3. *Let M and M^* be null parallel p -equidistant B -scrolls in R_1^3 .*

i) If K and K^ denote Gaussian curvatures of M and M^* , respectively. In this case, there is following relation between K and K^* :*

$$K^* = K \left(\frac{ds}{ds^*}\right)^2.$$

ii) The mean curvature of M and M^ is denoted by H and H^* , respectively. Therefore, there is a relation between H and H^* as follows:*

$$H^* = H \left(\frac{ds}{ds^*}\right).$$

Proof: i) To calculate the Gaussian curvatures K and K^* of M and M^* , respectively, considering the equations (1.12), (2.16) and (2.17) we have

$$(2.19) \quad K = \tau^2$$

and

$$(2.20) \quad K^* = \tau^{*2}.$$

From equations (2.10), (2.19) and (2.20), we obtain the relation between K and K^* as,

$$(2.21) \quad K^* = K \left(\frac{ds}{ds^*}\right)^2.$$

ii) Now, we compute the mean curvatures H and H^* of M and M^* , respectively. Using the equations (1.13), (2.16) and (2.17), it is easy to see that the mean curvatures H and H^* are given by

$$(2.22) \quad H = -\tau$$

and

$$(2.23) \quad H^* = -\tau^*$$

respectively. Considering the equations (2.10) and (2.22) together with the last equation, it is clear that

$$(2.24) \quad H^* = H \left(\frac{ds}{ds^*}\right).$$

Theorem 2.4. In R_1^3 , let M and M^* be null parallel p -equidistant B -scrolls and k_i and k_i^* , $1 \leq i \leq 2$ be the principal curvatures of M and M^* , respectively. In this case, the relations between k_i and k_i^* are as,

$$k_i^* = k_i \left(\frac{ds}{ds^*} \right), 1 \leq i \leq 2.$$

Proof: Since the principal curvatures of M are the roots of the characteristic polynomial which is given by " $\det(S - kI_2) = 0$ ", we have

$$(2.25) \quad k_1 = k_2 = -\tau.$$

Similarly, the principal curvatures of M^* are

$$(2.26) \quad k_1^* = k_2^* = -\tau^*.$$

Thus, from the equation (2.10) together with the equations (2.25) and (2.26), we reach

$$(2.27) \quad k_i^* = k_i \left(\frac{ds}{ds^*} \right), \quad 1 \leq i \leq 2$$

Theorem 2.5. In R_1^3 , let us assume that M and M^* are non-developable null parallel p -equidistant B -scrolls, λ and λ^* are the distribution parameters of M and M^* , respectively. Then the following relation satisfy

$$\lambda^* = \lambda \left(\frac{ds^*}{ds} \right).$$

Proof: Taking into consideration the equations (1.10) and (2.10) the following relation can be obtained between the distribution parameters λ and λ^* of M and M^* , respectively.

$$(2.28) \quad \lambda^* = \frac{1}{\tau^*} = \frac{1}{\tau} \frac{ds^*}{ds} = \lambda \frac{ds^*}{ds}.$$

Theorem 2.6. Let M and M^* be non-developable null parallel p -equidistant B -scrolls in R_1^3 .

i) There is the relation between the second Gaussian curvatures K_{II} and K_{II}^* of M and M^* as follows

$$K_{II}^* = K_{II} \left(\frac{ds}{ds^*} \right)$$

ii) There is the following relation between H_{II} and H_{II}^* ,

$$H_{II}^* = H_{II} \left(\frac{ds}{ds^*} \right)$$

where H_{II} and H_{II}^* are the second mean curvatures of M and M^* , respectively.

Proof: Suppose that M and M^* are non-developable null parallel p -equidistant B -scrolls in R_1^3 . Now we compute the coefficients of the first fundamental form E , F and G and the second fundamental form e , f and g of M . Using the equations (1.14), (1.15) and (2.2), we obtain that

$$(2.29) \quad E = v^2 \tau^2, \quad F = -1, \quad G = 0$$

Also

$$(2.30) \quad e = \kappa(s) - v^2 \tau^3, \quad f = \tau, \quad g = 0.$$

Following similar way, the first and second fundamental forms of non-developable null parallel B-scroll M^* are as

$$(2.31) \quad E^* = v^{*2}, \quad F^* = -1, \quad G^* = 0$$

and

$$(2.32) \quad e^* = \kappa^* (s^*) - v^{*2} \tau^{*3}, \quad f^* = \tau^*, \quad g^* = 0.$$

respectively.

i) Substituting e, f, g into the second Gaussian curvature matrix form, K_{II} which is given by equation (1.16), we get

$$K_{II} = -\tau$$

Similarly, we see that

$$K_{II}^* = -\tau^*.$$

From the equation (2.10) and the last two equations, it is easily to see that

$$(2.33) \quad K_{II}^* = K_{II} \left(\frac{ds}{ds^*} \right).$$

ii) Taking into consideration the equations (2.30), (2.32), (1.12) and (1.13) with the equation (1.17), we obtain

$$H_{II} = H$$

and

$$H_{II}^* = H^*.$$

From theorem 2.3, we have

$$H_{II}^* = H_{II} \left(\frac{ds}{ds^*} \right)$$

The results in the study are confirmed by the following example.

Example 2.1. In R_1^3 , let us assume that the null parallel p -equidistant B-scrolls M and M^* are parametrically given by

$$\varphi(s, v) = (s, \cos s, \sin s) + v \left(\frac{1}{2}, \frac{1}{2} \sin s, -\frac{1}{2} \cos s \right)$$

and

$$\varphi^*(s^*, v^*) = (s^* + 2, \cos s^* + 2, \sin s^* + 2) + v^* \left(\frac{1}{2}, \frac{1}{2} \sin s^*, -\frac{1}{2} \cos s^* \right)$$

respectively (Figure 1 and Figure 2). In this case, using the equations (1.6) and (2.5) we find

$$\kappa^* = \kappa = 1, \quad \tau^* = \tau = -\frac{1}{2},$$

Thus, substituting the last equations into the equations (2.16) and (2.17), the matrices corresponding to shape operators of M and M^* are to be

$$S^* = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix} = S$$

In addition to that, from the equations (1.16) and (1.17), there is a relation between the second Gauss curvature and the second mean curvature as follows, respectively

$$K_{II}^* = K_{II} = \frac{1}{2}, \quad H_{II}^* = H_{II} = \frac{1}{2}.$$

Lastly, considering the equations (2.16) and (2.17), the principal curvatures of M and M^* are as

$$k_i^* = k_i = 1, \quad 1 \leq i \leq 2$$

Figure 1. $\varphi(s, v) = (e \cos s \sin s) + v(\frac{1}{2} \sin s - \frac{1}{2} \cos s)$ null parallel p -equidistant B-scroll

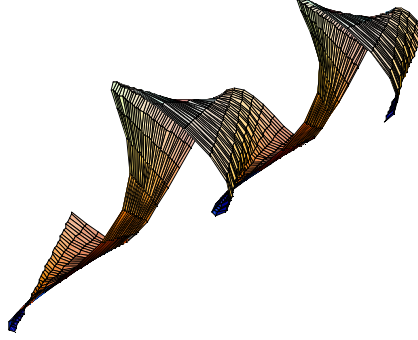
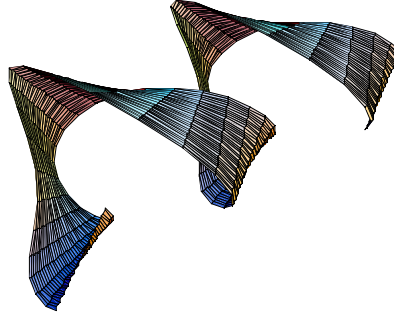


Figure 2. $\varphi^*(s^*, v^*) = (s^* + 2, \cos s^* + 2, \sin s^* + 2) + v^*(\frac{1}{2}, \frac{1}{2} \sin s^*, -\frac{1}{2} \cos s^*)$ null parallel p -equidis



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